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LAMINAR FORCED CONVECTION IN ECCENTRIC ANNULI*

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Abstract—Forced convection to hydrodynamically and thermally fully developed laminar flow in eccentric annuli is studied. Following Reynolds *et al.* [1], we determine the solutions of the energy equation which satisfy certain fundamental boundary conditions. These fundamental solutions can be superposed to satisfy a wide variety of boundary conditions. Exact solutions of the energy equation could not be found, so an approximate solution was determined. Nusselt numbers, wall temperature and heat fluxes, wall-fluid temperature differences, and mean fluid temperatures are presented for a wide range of eccentricities and radius ratios. It is believed that these results are accurate to within 1 per cent.

NOMENCLATURE ‡

c_p, fluid specific heat $[L^2/t^2T]$;
D_h, hydraulic diameter, $2(R_2 - R_1)$ [L];
E, absolute eccentricity [L];
e, dimensionless eccentricity, defined in equation (3d);
$h_{lj}^{(k)}$, average heat-transfer coefficient on wall l for fundametal problem of k'th kind when the nonzero boundary condition is applied to wall j, defined equation (36) $[M/t^3T]$;
$\boldsymbol{i}_r$, unit vector in r direction;
$\boldsymbol{i}_\phi$, unit vector in ϕ direction;
k, thermal conductivity of fluid $[ML/t^3T]$;
$\boldsymbol{n}$, unit normal to outer boundary, defined in equation (19);
$Nu_{lj}^{(k)}$, Nusselt number based on $h_{lj}^{(k)}$ defined in equation (37);
p, fluid pressure $[M/Lt^2]$;
q_j'', heat flux at wall j $[M/t^3]$;
R, radial coordinate [L];
R_1, radius of inner boundary of annulus [L];
R_2, radius of outer boundary of annulus [L];
r, dimensionless radial coordinate, defined in equation (3a);
T^*, fluid temperature [T];
$T_j^{(k)}$, dimensionless fluid temperature for the k'th fundamental problem when the nonzero boundary condition is applied to wall j.
$T_{lj}^{(k)}$, dimensionless average temperature of wall l for the k'th fundamental problem when the nonzero boundary condition is applied to wall j;
$T_{mj}^{(k)}$, dimensionless fluid cup mixing temperature for k'th fundamental problem when nonzero boundary condition is applied to wall j, defined in equation (42);
u^*, fluid velocity [L/t];
$\bar{u}^*$, average fluid velocity [L/t];
u, dimensionless velocity, defined in equation (3b);

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‡ Any consistent set of units may be used. Dimensions are given in terms of mass (M), length (L), time (t), and temperature (T).

W, dimensionless flow rate parameter, defined in equation (20);
w, fluid flow rate [M/t];
z, axial coordinate [L];
α, radius ratio, R_1/R_2;
ϵ, dimensionless eccentricity, defined in equation (3e);
μ, fluid viscosity [M/Lt];
ρ, fluid density [M/L^3];
Γ, outer boundary of annulus;
$\Phi_{lj}^{(k)}$, dimensionless flux on wall l for the k'th fundamental problem when the nonzero boundary condition is applied to wall j;
ϕ, angular coordinate.

I. INTRODUCTION

ECCENTRIC annuli are employed in a variety of heat transfer systems. Since an annulus contains two surfaces on which thermal conditions may be independently specified, there are a large number of heat transfer problems of significant interest. Some years ago, Reynolds *et al.* [1] completely solved the problem of heat transfer to fully developed laminar flow in concentric annuli. They realized that with the usual assumptions the energy equation is linear, and therefore that a temperature field satisfying arbitrary boundary conditions could be determined by simply adding appropriate multiples of temperature fields satisfying certain fundamental boundary conditions. Four fundamental boundary conditions were defined and the temperature fields satisfying these boundary conditions were called fundamental solutions.

We will be concerned with laminar forced convection in eccentric annuli, but we will limit our study to hydrodynamically and thermally fully developed flows. Fully developed fundamental solutions of the third kind result in indeterminate forms which cannot be evaluated. In this paper, we will obtain fully developed fundamental solutions of the first, second, and fourth kinds.

The only previously published study of laminar forced convection in eccentric annuli is that of Cheng and Hwang [2]. Their boundary conditions are not one of the fundament sets. We will discuss their solution in more detail at the appropriate place.

II. MATHEMATICAL DEVELOPMENT

Because of symmetry we need only consider the domain shown in Fig. 1. A cylindrical coordinate system is established with origin at the center of the small cylinder. The equation of the outer boundary is

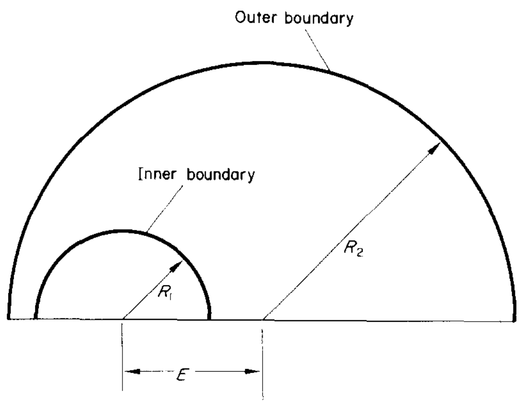

FIG. 1. Basic domain for an eccentric annulus.

$$\Gamma(R, \phi) \equiv R^2 - 2RE\cos\phi + E^2 - R_2^2 = 0. \quad (1)$$

We assume that the fluid is in fully developed laminar flow. With the additional assumption that fluid properties are independent of temperature the momentum balance becomes

$$\frac{1}{R}\frac{R}{\partial R}\left(R\frac{\partial u^*}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 u^*}{\partial \phi^2} = \frac{1}{\mu}\frac{\partial p}{\partial z}. \quad (2)$$

It is convenient to introduce the following dimensionless variables:

$$r \equiv R/R_1, \quad (3a)$$

$$u \equiv u^* \bigg/ \left(-\frac{R_1^2}{\mu}\frac{\partial p}{\partial z}\right), \quad (3b)$$

$$\alpha \equiv R_1/R_2, \tag{3c}$$

$$e \equiv E/R_1, \tag{3d}$$

$$\epsilon \equiv E/(R_2 - R_1) = \alpha e/(1 - \alpha). \tag{3e}$$

In terms of these dimensionless variables the equation for the outer boundary becomes

$$\Gamma(r, \phi) \equiv r^2 - 2re\cos\phi + e^2 - 1/\alpha^2 = 0, \tag{4}$$

and the momentum balance becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = -1. \tag{5}$$

Equation (5) is to be solved subject to the following boundary conditions: Symmetry conditions require

$$\frac{\partial u}{\partial \phi}(r, 0) = \frac{\partial u}{\partial \phi}(r, \pi) = 0. \tag{6}$$

Along solid boundaries the velocity vanishes

$$u(1, \phi) = 0, \tag{7a}$$

$$u(\Gamma) = 0. \tag{7b}$$

By the use of bipolar coordinates, it is possible to obtain an exact solution for the velocity field. It does not seem possible, however, to use the same technique to obtain an exact solution for the temperature field. Since we must use an approximate technique to determine the temperature field, we find it simpler and more convenient to use the same approximate technique to obtain both the temperature and velocity fields. The general solution of equation (5) which satisfies boundary conditions (6) and (7a) is

$$u(r, \phi) = \tfrac{1}{4}(1 - r^2) + B_0 \ln r + \sum_{n=1}^{\infty} (r^n - r^{-n}) \times B_n \cos n\phi. \tag{8}$$

It is impossible to choose the B's so that boundary conditions (7b) is satisfied everywhere. We therefore use a technique in which boundary condition (7b) is approximately satisfied. Since the same technique is used to solve the temperature equation, we will postpone a more detailed description of the method until after the temperature equation is derived.

With the usual assumptions that fluid properties are independent of temperature and that axial conduction is negligible, the energy balance becomes

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T^*}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 T^*}{\partial \phi^2} = \frac{\rho u c_p}{k}\frac{\partial T^*}{\partial z}. \tag{9}$$

Symmetry considerations give two boundary conditions on temperature:

$$\frac{\partial T^*}{\partial \phi}(r, 0) = \frac{\partial T^*}{\partial \phi}(r, \pi) = 0. \tag{10}$$

Additional boundary conditions on temperature depend on which of the fundamental problems is under discussion. For the fundamental problem of the first kind, we specify that the inner and outer surfaces are kept at uniform, but different, temperatures.† Denote the temperatures at the inner and outer surfaces by T_i^* and T_0^*. We introduce the dimensionless temperature $T_0^{(1)}$, defined by

$$T_0^{(1)} \equiv \frac{T^* - T_i^*}{T_0^* - T_i^*}. \tag{11}$$

In equation (11) we have adopted the notation used by Reynolds *et al.* [1], in which $T_j^{(k)}$ is the dimensionless temperature for the fundamental problem of the k'th kind when the nonzero boundary condition is applied to wall j. The boundary conditions on $T_0^{(1)}$ are then

$$T_0^{(1)}(1) = 0, \tag{12a}$$

$$T_0^{(1)}(\Gamma) = 1. \tag{12b}$$

With the two walls held at different temperatures, it is clear that under fully developed conditions whatever heat enters at one wall leaves at the

† For the fully developed case such as we are considering here, there is essentially only one fundamental problem of the first kind. This is not the situation in the thermal development region [1], where there are two distinct problems depending on whether it is the inside or outside surface which is raised above the entering fluid temperature.

other, so that $\partial T^*/\partial z$ is zero. For the fundamental problem of the first kind, then, in terms of dimensionless variables, equation (9) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_0^{(1)}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T_0^{(1)}}{\partial \phi^2} = 0. \quad (13)$$

For the fundamental problem of the second kind we specify that at wall j there is a uniform axial heat addition, q_j'' Btu/h ft^2, while the other wall is insulated. For the concentric annulus these two conditions completely specify the problem. In contrast, in an eccentric annulus we must further specify the peripheral variation of temperature or flux on the active wall. Two limiting cases may be distinguished: uniform temperature and uniform flux. Although both conditions are of interest, the concept of fundamental solutions which may be added to obtain solutions of more complex problems requires that the flux be uniform peripherally as well as axially, and this is the case which we will examine (The problem studied by Cheng and Hwang [2] is related to the fundamental problem of the second kind, in which, however, there is heat addition at both walls, and in which the peripheral temperature is uniform.) We introduce the dimensionless temperature

$$T_j^{(2)} = \frac{T^*}{q_j'' D_h/k}. \quad (14)$$

In equation (14), j is either i or o depending on whether it is the inner or outer wall which is active. When the inner wall is active the boundary conditions on $T_i^{(2)}$ are

$$\left.\frac{\partial T_i^{(2)}}{\partial r}\right|_{r=1} = -\frac{\alpha}{2(1-\alpha)}. \quad (15a)$$

$$\text{On } \Gamma: \frac{\partial T_i^{(2)}}{\partial n} = 0. \quad (15b)$$

In equation (15b) the derivative is taken normal to the outer boundary. It is most conveniently calculated using the vector identity

$$\frac{\partial T_i^{(2)}}{\partial n} = \boldsymbol{n}\cdot\nabla T_i^{(2)}$$

where $\boldsymbol{n}$ is the unit outward normal on the outer boundary. Equation (15b) therefore becomes

$$\text{On } \Gamma: \boldsymbol{n}\cdot\nabla T_i^{(2)} = 0. \quad (15c)$$

In turn, $\boldsymbol{n}$ is calculated by

$$\boldsymbol{n} = \frac{\nabla\Gamma}{|\nabla\Gamma|}.$$

Introducing equation (4) into this equation, we find

$$\boldsymbol{n} = \frac{(r - e\cos\phi)\,\boldsymbol{i}_r + (e\sin\phi)\,\boldsymbol{i}_\phi}{[(r - e\cos\phi)^2 + (e\sin\phi)^2]^{1/2}}. \quad (16)$$

When the outer wall is active the boundary conditions on $T_0^{(2)}$ are

$$\left.\frac{\partial T_0^{(2)}}{\partial r}\right|_{r=1} = 0. \quad (17a)$$

$$\text{On } \Gamma: \boldsymbol{n}\cdot\nabla T_0^{(2)} = \frac{\alpha}{2(1-\alpha)}. \quad (17b)$$

For the fully developed case, $\partial T^*/\partial z$ is constant and may be evaluated by an energy balance. When the inner wall is active, equation (9) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_i^{(2)}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T_i^{(2)}}{\partial \phi^2} = \frac{\pi\alpha u}{2(1-\alpha)W}. \quad (18)$$

When the outer wall is active, equation (9) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_0^{(2)}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T_0^{(2)}}{\partial \phi^2} = \frac{\pi u}{2(1-\alpha)W}. \quad (19)$$

In these equations W is defined by the double integral

$$W = \int_0^\pi \mathrm{d}\phi \int_1^\Gamma ur\,\mathrm{d}r. \quad (20)$$

The upper limit on the first integral indicates that the integration is to extend to the outer boundary.

For the fundamental problem of the fourth kind, we specify that at one wall there is a uniform axial heat addition, q_j'' Btu/h ft^2, while the other wall is maintained at a constant temperature, T_e^*. We introduce the dimensionless temperature

$$T_j^{(4)} \equiv \frac{T^* - T_e^*}{q_j'' D_h / k}. \tag{21}$$

When the inner wall is active the boundary conditions on $T_i^{(4)}$ are

$$\left.\frac{\partial T_i^{(4)}}{\partial r}\right|_{r=1} = -\frac{\alpha}{2(1-\alpha)}, \tag{22a}$$

$$T_i^{(4)}(\Gamma) = 0. \tag{22b}$$

When the outer wall is active the boundary conditions on $T_0^{(4)}$ are

$$T_0^{(4)}(1, \phi) = 0, \tag{23a}$$

$$\text{On } \Gamma: \boldsymbol{n} \cdot \nabla T_0^{(4)} = \frac{\alpha}{2(1-\alpha)}. \tag{23b}$$

As was the case for the fundamental problem of the first kind, for the fundamental problem of the fourth kind all the heat which enters at one wall leaves at the other, so that $\partial T^*/\partial z$ is zero. The energy equation satisfied by $T_j^{(4)}$ is therefore the simple form, equation (13).

With the equations and boundary conditions established we may now determine the temperature field. Since the procedure followed in all cases is similar, we will conserve space and report in detail only the solution for the fundamental problem of the second kind with the inner wall active. We must solve equation (18) subject to boundary conditions (10) and (15). The solution of equation (18) which satisfies boundary conditions (10) and (15a) is

$$T_i^{(2)}(r, \phi) = \frac{\pi\alpha}{8(1-\alpha)W}\left\{\frac{r^2}{4} - \frac{r^4}{16} - \frac{\ln r}{4} + B_0\left[r^2(\ln r - 1) + \ln r\right] + B_1\left[\frac{r^3}{2} - 2r\ln r - \frac{1}{2r}\right]\cos\phi + \sum_{n=2}^{\infty} B_n\left[\frac{r^{n+2}}{n+1} + \frac{r^{-n+2}}{n-1} + \frac{2r^{-n}}{(n+1)(n-1)}\right]\cos n\phi\right\} - \frac{\alpha}{2(1-\alpha)}\ln r + C_0 + \sum_{n=1}^{\infty} D_n(r^{-n} + r^n)\cos n\phi. \tag{24}$$

In equation (24) we may arbitrarily set C_0 equal to zero, since, as the boundary conditions show, we are solving a Neumann problem, and therefore cannot determine the absolute temperature level. As in the case of the B's and equation (7b), we find that it is not possible to choose the D's so that equation (15b) is satisfied everywhere. We therefore use an approximate method in which we truncate the series appearing in equations (8) and (24) and solve for the unknown B's and D's by satisfying equations (7b) and (15b) at a finite number of points. We truncate the B series in equation (8) at $n = N - 1$, and the D series in equation (24) at $n = N$, so that there are a total of N unknown B's and N unknown D's. Let M denote the number of points on the outer boundary at which (7b) and (15b) are satisfied.

We could set $M = N$ and solve for the B's D's. This in fact is just the collocation method [3]. However, the collocation method often leads to difficulties. For example, in a study of laminar flow in eccentric annuli, Tiedt [4] reported that at an eccentricity of 0·8 he could not obtain solutions for radius ratios smaller than 0·2; and at an eccentricity of 1, he could not obtain solutions for radius ratios smaller than 0·4. We decided, therefore, to use the least squares method [5] which was first used for heat transfer problems by Sparrow [6]. In the least squares method, the number of points at which the boundary condition is satisfied is much larger than the number of unknowns. We found that choosing $M = 3N$ was satisfactory.

To avoid the necessity of using weight factors [5], we selected the M points to be uniformly distributed on Γ. A simple trigonometric analysis of Fig. 1 shows that the M points will be uniformly distributed on Γ if ϕ coordinates are calculated using the formula

$$\phi_j = \arctan\left[\frac{\sin\left(\frac{j\pi}{M-1}\right)}{\alpha e + \cos\left(\frac{j\pi}{M-1}\right)}\right],$$

$$j = 0, 1, \ldots, M-1. \qquad (25)$$

In using equation (25) we always choose $0 \leqslant \phi \leqslant \pi$. The r_j coordinates are calculated from the solution of equation (4). Satisfying equation (7b) gives the following equations for the B's

$$B_0 \ln r_j + \sum_{n=1}^{N-1} (r_j^n - r_j^{-n}) B_n \cos n\,\phi_j = \tfrac{1}{4}(r_j^2 - 1), \; j = 0, 1, \ldots, M-1. \qquad (26)$$

Similarly, satisfying equation (15b) gives the following equations for the D's.

$$\sum_{n=1}^{N} D_n n[f_1(r_j^{n-1} - r_j^{-n-1}) \cos n\phi_j - f_2(r_j^{-n} + r_j^n) \sin n\,\phi_j] = \frac{\pi\alpha}{8(1-\alpha)W} \times \Bigg(-f_1\Bigg\{\frac{r_j}{2} - \frac{r_j^3}{4} - \frac{1}{4r_j} + B_0\left[r_j(2\ln r_j - 1) + \frac{1}{r_j}\right] + B_1\left[\tfrac{3}{2}r_j^2 - 2(\ln r_j + 1) + \frac{1}{2r_j^2}\right]\cos\phi_j$$
$$+ \sum_{n=2}^{N-1} B_n\left[\frac{n+2}{n-1} r_j^{n+1} - \frac{n-2}{n-1} r^{-n+1} - \frac{2n}{(n-1)(n+1)} r^{-(n+1)}\right]\cos n\phi_j\Bigg\} + f_2\Bigg\{B_1\left[\frac{r_j^3}{2} - 2r_j \ln r_j - \frac{1}{2r_j}\right]\sin\phi_j$$
$$+ \sum_{n=2}^{N-1} B_n n\left[\frac{r^{n+2}}{n+1} + \frac{r_j^{-n+2}}{n-1} + \frac{2r^{-n}}{(n+1)(n-1)}\right] \sin n\phi_j\Bigg\}\Bigg) + \frac{f_1\alpha}{2(1-\alpha)r_j}, \qquad j = 0, 1, \ldots, M-1 \qquad (27)$$

In equation (27) we have introduced f_1 and f_2 which are defined as

$$f_1 \equiv r_j^2 - r_j e \cos\phi_j$$
$$f_2 \equiv e \sin\phi_j.$$

We first solve equation (26) for the B's, and then, with the B's known we solve equation (27) for the D's. The best way of solving equations (26) and (27) is by the use of the Gram–Schmidt orthogonalization procedure described by Davis [5].

Essentially the same procedure is used to develop the other fundamental solutions, but the equations for the temperature and the D's are, of course, different.

III. RESULTS

It is important to estimate the accuracy of the results obtained using this approximate method. For the velocity field there is an exact solution against which we can compare our approximate solution. Tiedt [4] tabulates the product of friction factor and Reynolds number as a function of radius ratio, α, and eccentricity, ϵ. The friction factor and Reynolds number are defined as

$$\lambda \equiv \left(-\frac{dp}{dz}\right)\frac{2D_h}{\rho(\bar{u}^*)^2} \qquad (28)$$

$$Re \equiv \frac{D_h \bar{u}^* \rho}{\mu}. \qquad (29)$$

Introducing dimensionless variables and using equation (20) we find that the product of friction factor and Reynolds number is

$$\lambda Re = \frac{4\pi(1-\alpha^2)(1-\alpha)^2}{W\alpha^2}. \qquad (30)$$

With the B's known from the solution of equation (26), the velocity field may be calculated as given by equation (8), and W may be determined by numerically evaluating the double integral given by equation (20). We found that an 8 by 8 Gaussian integration formula was sufficient to maintain the error in W below 0·1 per cent. With W known, equation (30) may be used to calculate λRe. Using only five unknowns and

Table 1. Comparison of exact and approximate values of λRe

α	Eccentricity					
	0·8			1·0		
	Exact	Approximate	% Error	Error	Approximate	% Error
0·2	56·724	56·721	0·005	48·736	48·716	0·04
0·1	60·523	60·481	0·07	54·402	54·367	0·06
0·02	64·727	64·654	0·11	61·598	61·638	−0·06
0·01	65·236	65·170	0·10	62·761	67·788	−0·04
0·005	65·408	65·350	0·09	63·370	63·385	−0·02

requiring the velocity to vanish on 15 points on the outer boundary, approximate values of λRe were calculated and are compared with Tiedt's exact values in Table 1. Comparisons are made at combinations of α and ε for which Tiedt found the collocation method failed, and for which he used the exact solution of Piercy *et al.* [7]. Table 1 shows that even for these difficult geometries the maximum error in the least squares results is 0·1 per cent. For smaller values of ϵ and larger values of α the error would be less. For all ϵ, in the limit as $\alpha \to 0$, the core vanishes and the annulus becomes a cylindrical tube, for which we have $\lambda Re = 64$. We conclude, therefore, that the least squares method predicts accurate values of λRe over the complete range $0 \leqslant \epsilon \leqslant 1, 0 \leqslant \alpha \leqslant 1$.

Of course, there are no exact heat transfer results against which to compare our results, otherwise there would be no need for this study. However, there are a number of ways we can estimate the accuracy of our results. One method depends on the fact that the energy equation is elliptic. For elliptic equations the maximum principle states that the maximum deviation between the approximate and exact solutions in the interior of a region is no larger than the maximum deviation between the approximate and exact solutions on the boundary. For the fundamental problems of the first kind and of the fourth kind when the inner wall is active, we specify the temperature along the outer boundary. After the D's are determined it is a simple matter to calculate the temperatures along the boundary and to determine the maximum deviation between the calculated and and specified temperatures. As we shall explain below, our primary interest is in average quantities, e.g. the average Nusselt number and the difference between the average wall temperature and the average fluid temperature, and it is therefore perhaps more meaningful to calculate the deviation between the average wall temperature and the specified wall temperature. The average error is then calculated as the ratio of this average deviation to the difference betwen the average fluid and wall temperatures. For the fundamental problem of the first kind, we we found that the average error is less than 1 per cent for all values of α if ϵ is less than or equal to 0·97. For the fundamental problem of the fourth kind with the inner wall active, we found the average error is less than 1 per cent in the region defined by the inequalities:

$$\epsilon \leqslant 0{\cdot}9, \qquad 0{\cdot}0 \leqslant \alpha \leqslant 1{\cdot}0$$

$$\epsilon > 0{\cdot}9, \qquad 0{\cdot}2 \leqslant \alpha \leqslant 1{\cdot}0.$$

This method of estimating the error cannot be used for the fundamental problem of the second kind or for the foundamental problem of the fourth kind with the outer wall active, since for these problems we specify the gradient of temperature and not the temperature itself. For these cases, we estimated the error by repeating a particular calculation with increasing values of M and N until further increases caused no significant change in the results or until the equations for the D's became ill-conditioned and no solution was obtained. For the fundamental problem of the second kind with the outer wall active we determined that the errors in the results are less than 1 per cent for all radius ratios if the eccentricity is less than or equal to 0·98. For the fundamental problem of the second kind with the inner wall active we found that the region in which accurate results were obtained was slightly smaller. The simplest way to define the region in which accurate results were obtained is to give results in the Tables and Figures which follow only for those combinations of α and ϵ for which the errors are estimated to be less than 1 per cent. A similar procedure was used for the fundamental problem of the fourth kind with the inner wall active. Most of the results reported here were obtained using N equal to 15 and M equal to 45. A typical calculation took about $\frac{3}{4}$-s on a CDC-6600.

In presenting our results it is convenient and useful to follow Reynolds *et al.* [1]. For the fundamental problem of the first kind they define the dimensionless fluxes

$$\Phi_{ii}^{(1)} = -\Phi_{i0}^{(1)} = -\frac{D_h}{T_i^* - T_0^*}\overline{\left(\frac{\partial T^*}{\partial R}\right)}_{R=R_1},$$

$$\Phi_{00}^{(1)} = -\Phi_{0i}^{(1)} = -\frac{D_h}{T_i^* - T_0}\overline{\left(\frac{\partial T^*}{\partial n}\right)}_{\Gamma}.$$

In terms of dimensioness variables these equations become

$$\Phi_{ii}^{(1)} = -\Phi_{i0}^{(1)} = -\frac{2(1-\alpha)}{\alpha}\overline{\left(\frac{\partial T}{\partial r}\right)}_{r=1}, \quad (31)$$

$$\Phi_{00}^{(1)} = -\Phi_{0i}^{(1)} = -\frac{2(1-\alpha)}{\alpha}\overline{\left(\frac{\partial T}{\partial n}\right)}_{\Gamma}. \quad (32)$$

In these equations an overbar indicates an average value. We have again adopted the nomenclature of Reynolds *et al.* [1]. $\Phi_{lj}^{(k)}$ indicates the dimensionless flux at wall l for the fundamental problem of the k'th kind when the nonzero boundary condition is applied at wall j. It is not necessary to use equation (32) to calculate $\Phi_{0i}^{(1)}$; since the heat which enters at one wall leaves at the other, we have the simple relationship

$$\Phi_{0i}^{(1)} = -\Phi_{00}^{(1)} = -\alpha\Phi_{ii}^{(1)}. \quad (33)$$

For the fundamental problem of the second kind, Reynolds *et al.* define the dimensionless flux so that it is unity on the active wall and zero on the adiabatic wall. Similarly, for the fundamental problem of the fourth kind the dimensionless flux is defined so that it is unity at the wall on which the flux is specified. Straightforward energy balence considerations, like those which lead to equation (33), give

$$\Phi_{0i}^{(4)} = -\alpha, \quad (34)$$

$$\Phi_{i0}^{(4)} = -1/\alpha. \quad (35)$$

The average heat-transfer coefficient on wall l for the fundamental problem of the k'th kind when the nonzero boundary condition is applied to wall j is defined as

$$h_{lj}^{(k)} \equiv \frac{\overline{q_l''^{(k)}}}{T_{lj}^{*(k)} - T_{mj}^{*(k)}}, \quad (36)$$

and the average Nusselt number

$$Nu_{lj}^{(k)} \equiv \frac{D_h h_{lj}^{(k)}}{k}. \quad (37)$$

In equation (36), the overbar indicates an average value, $T_{lj}^{*(k)}$ is the average temperature on the l-th wall, and $T_{mj}^{*(k)}$ is the cup mixing fluid temperature. The evaluation of these average temperatures will be discussed shortly. For the fundamental problem of the first kind, introducing equation (36) into equation (37) and using equations (11), (31) and (32) gives

$$Nu_{lj}^{(1)} = \frac{\Phi_{lj}^{(1)}}{T_{lj}^{(1)} - T_{mj}^{(1)}}. \quad (38)$$

In the fully developed region we have

$$Nu_{0i}^{(1)} = Nu_{00}^{(1)}$$

$$Nu_{i0}^{(1)} = Nu_{ii}^{(1)}.$$

Similarly, for the fundamental problem of the second kind we have

$$Nu_{lj}^{(2)} = \begin{cases} \dfrac{1}{T_{jj}^{(2)} - T_{mj}^{(2)}}, & l = j \quad (39a) \\ 0 & l \neq j. \quad (39b) \end{cases}$$

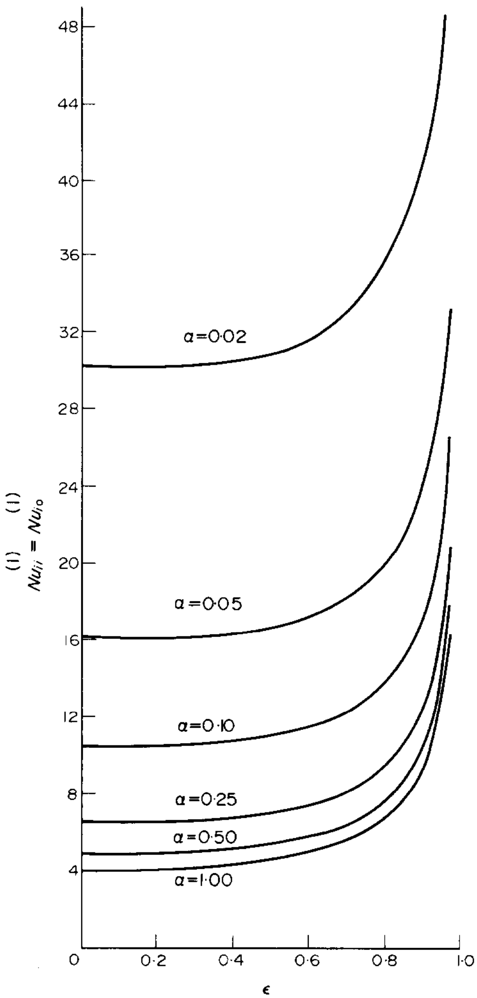

FIG. 2. Effect of eccentricity and radius ratio on $Nu_{ii}^{(1)}$ and $Nu_{io}^{(1)}$.

Finally, for the fundamental problem of the fourth kind we have

$$Nu_{ii}^{(4)} = \frac{1}{T_{ii}^{(4)} - T_{mi}^{(4)}} \tag{40a}$$

$$Nu_{0i}^{(4)} = \frac{\alpha}{T_{mi}^{(4)}} \tag{40b}$$

$$Nu_{00}^{(4)} = \frac{1}{T_{00}^{(4)} - T_{m0}^{(4)}} \tag{40c}$$

$$Nu^{(4)} = \frac{1}{\alpha T_{m0}^{(4)}}. \tag{40d}$$

The calculation of the average temperature on the inner wall presents no difficulty, but the average temperature on the outer wall is not straightforward. The appropriate equation is

$$T_{0j}^{(k)} = \frac{\alpha}{\pi} \int_0^{\pi} T_j^{(k)}(\Gamma)\,[r^2 + (\mathrm{d}r/\mathrm{d}\phi)^2]^{1/2}\,\mathrm{d}\phi. \tag{41}$$

In evaluating the integral, r and $\mathrm{d}r/\mathrm{d}\phi$ are obtained from the solution of equation (4). The cup mixing fluid temperature is defined in

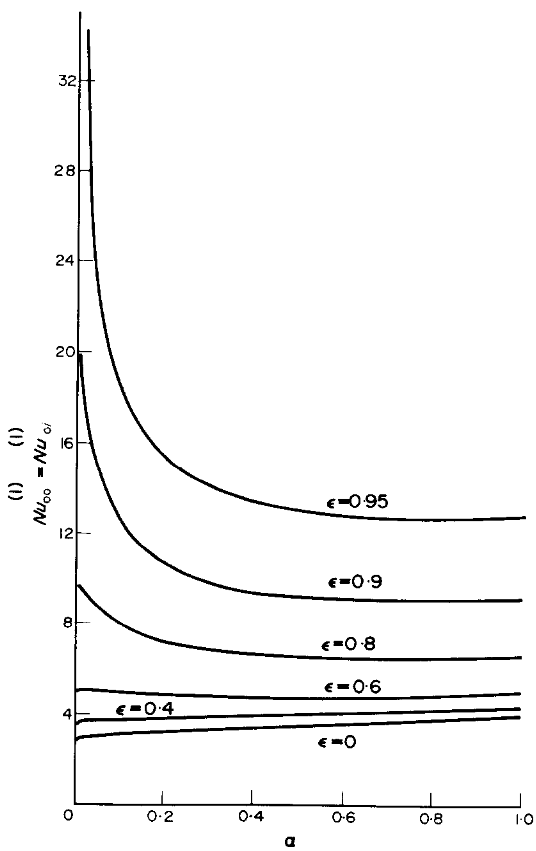

FIG. 3. Effect of eccentricity and radius ratio on $Nu_{00}^{(1)}$ and $Nu_{0i}^{(1)}$.

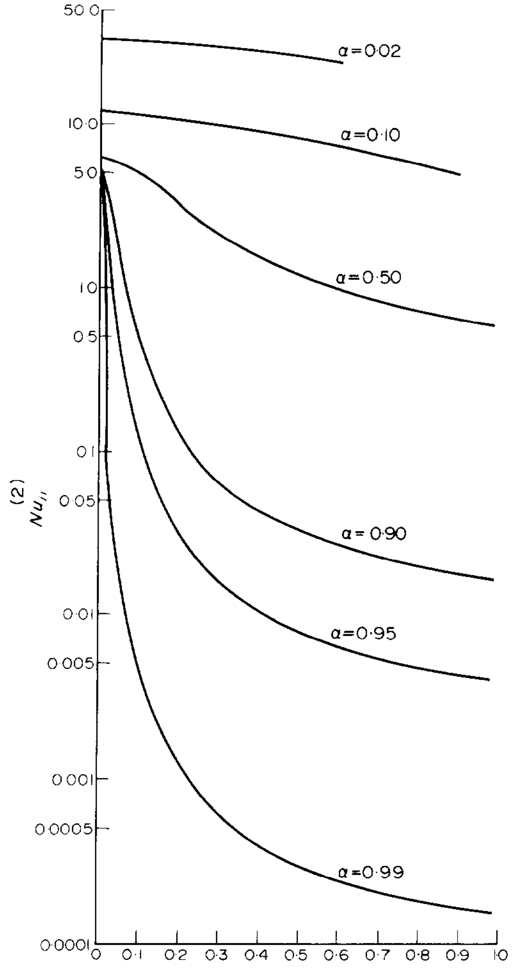

FIG. 4. Effect of eccentricity and radius ratio on $Nu_{ii}^{(2)}$.

the usual way and is calculated using the equation

$$T_{mj}^{(k)} = \frac{\int_0^\pi \mathrm{d}\phi \int_1^r ruT_j^{(k)}\,\mathrm{d}r}{W}. \tag{42}$$

The integrals in equations (41) and (42) must be evaluated numerically.

The various Nusselt numbers are shown as functions of radius ratio and eccentricity in Figs. 2–9. Results are shown only where they are believed to be accurate to within 1 per cent. We have tabulated various dimensionless fluxes and temperatures for eccentricities of 0·2, 0·4, 0·6, 0·8, 0·9 and 0·95.† Results for an eccentricity of zero are given by Reynolds *et al.* [1]. To conserve space, fluxes which may be calculated by simple equations as equations (33)–(35) were not tabulated. For concentric annuli there

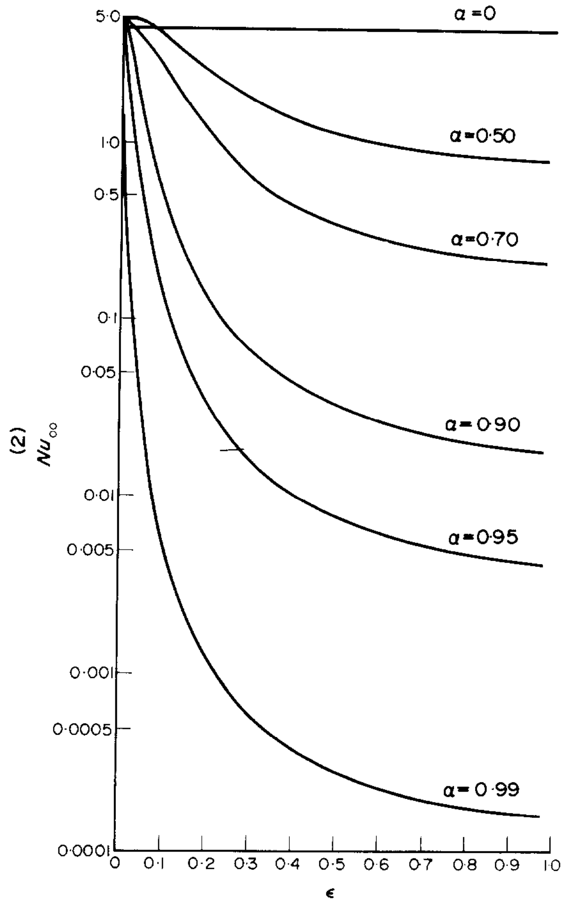

FIG. 5. Effect of eccentricity and radius ratio on $Nu_{oo}^{(2)}$.

† Tabular material is deposited as document NAPS 01185 with the ASIS National Auxiliary Publications Service, c/o CCM Information Corporation, 909 Third Avenue, New York, N.Y. 10022 and may be obtained for $2.00 for microfiche and $5.00 for photocopy.

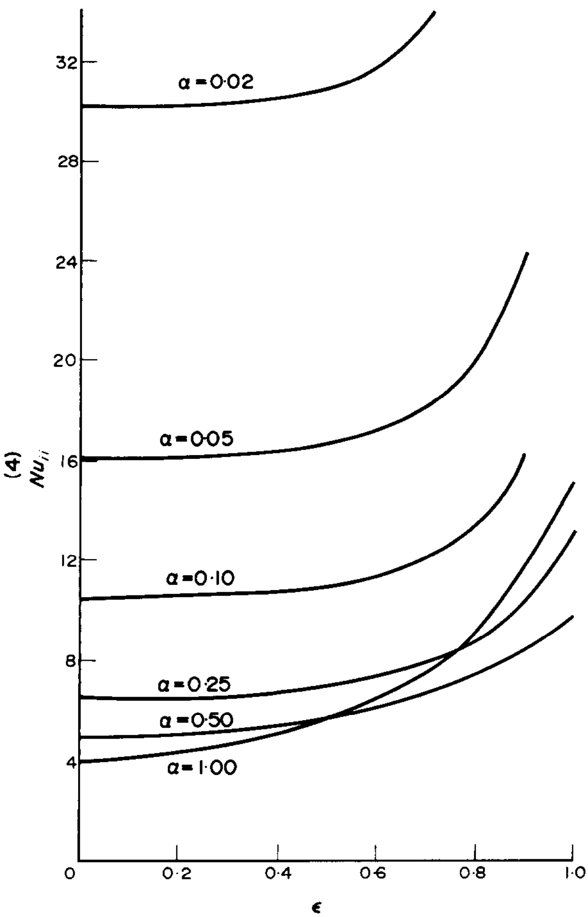

FIG. 6. Effect of eccentricity and radius ratio on $Nu_{ii}^{(4)}$.

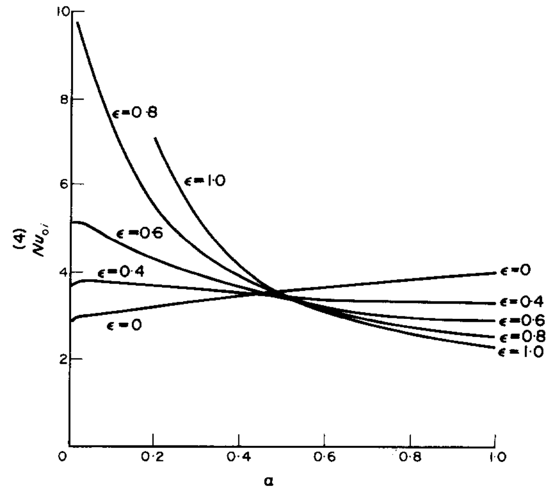

FIG. 7. Effect of eccentricity and radius ratio on $Nu_{0i}^{(4)}$.

are a number of simple relationships which the Nusselt numbers and dimensionless temperatures satisfy. For example, in concentric annuli

$$T_{ii}^{(4)} = \alpha T_{00}^{(4)}$$

$$Nu_{ii}^{(4)} = Nu_{i0}^{(4)}$$

$$Nu_{00}^{(4)} = Nu_{0i}^{(4)}$$

$$Nu_{ii}^{(1)} = Nu_{ii}^{(4)}$$

$$Nu_{00}^{(1)} = Nu_{00}^{(4)}.$$

Reference to the figures will show that none of these relationships is satisfied in eccentric annuli.

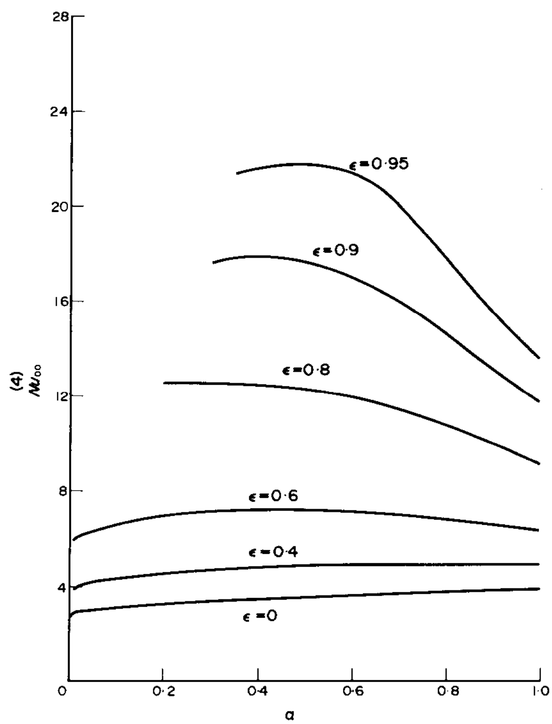

FIG. 8. Effect of eccentricity and radius ratio on $Nu_{00}^{(4)}$.

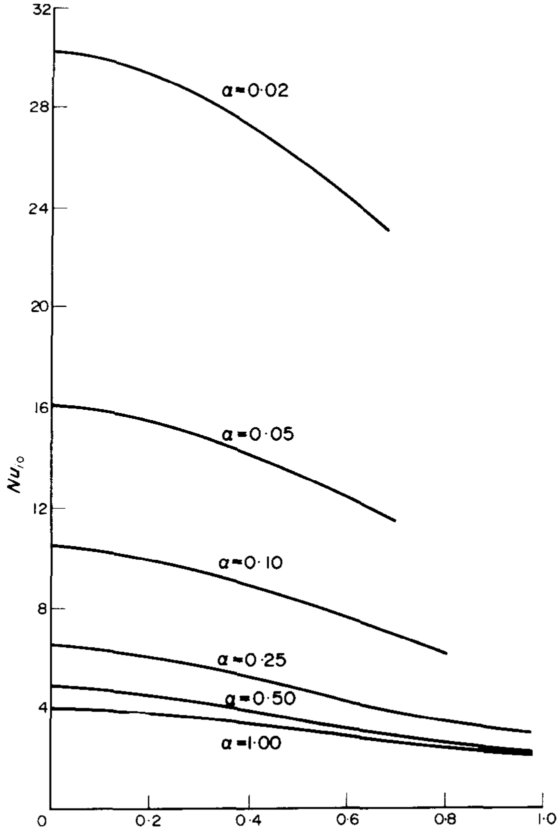

FIG. 9. Effect of eccentricity and radius ratio on $Nu_{io}^{(4)}$.

IV. SUMMARY

We have calculated fully developed solutions for the fundamental problems of the first, second, and fourth kinds in eccentric annuli. As shown by Reynolds *et al.* [1], these fundamental solutions may be superposed to give solutions to problems with a wide variety of boundary conditions. Since an exact solution of the temperature field equation could not be obtained, we used an approximate method in which a least squares technique is used to satisfy the boundary condition on the outside wall of the annulus. We found the least squares technique gives accurate solutions rapidly for almost all combinations of radius ratio and eccentricity.

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CONVECTION LAMINAIRE FORCÉE DANS DES ANNEAUX EXCENTRIQUES

Résumé—On étudie la convection forcée pour un écoulement laminaire entièrement développé hydrodynamiquement et thermiquement. En suivant Reynolds *et al.* nous déterminons les solutions de l'équation d'énergie qui satisfont certaines conditions aux limites fondamentales. On peut superposer ces solutions fondamentales afin de satisfaire une large variété de conditions aux limites. On n'a pu trouver de solutions exactes de l'équation d'énergie mais on a déterminé une solution approchée. On présente pour un large domaine d'excentricité et de rapports de rayon les nombres de Nusselt, les températures et flux thermiques pariétaux, les différences de température fluide-paroi et les températures moyennes du fluide. On pense que ces résultats ont une précision de 1 pour cent.

LAMINARE ERZWUNGENE KONVEKTION IN EXZENTRISCHEN RINGSPALTEN

Zusammenfassung—Die erzwungene Konvektion einer hydrodynamisch und thermisch vollausgebildeten laminaren Strömung in exzentrischen Ringspalten wurde untersucht. In Anlehnung an Reynolds und andere bestimmen wir die Lösungen der Energiegleichung, die gewissen grundlegenden Grenzbedingungen genügen. Diese Hauptlösungen kann man überlagern, um starken Veränderungen der Randbedingungen zu genügen. Exakte Lösungen der Energiegleichung konnten nicht gefunden werden. Statt dessen wurde eine Näherungslösung bestimmt. Nusselt-Zahlen, Wandtemperaturen und Wärmeströme, Temperaturdifferenzen zwischen Wand und Fluid und mittlere Fluidtemperaturen werden für einen weiten Bereich der Exzentrizitäten und Radiusverhältnisse dargestellt. Diese Ergebnisse dürften bis auf 1% genau sein.

ЛАМИНАРНАЯ ВЫНУЖДЕННАЯ КОНВЕКЦИЯ В ЭКСЦЕНТРИЧЕСКИХ КОЛЬЦЕВЫХ КАНАЛАХ

Аннотация—Использовалась вынужденная конвекция при гидродинамически и термически полностью развитом ламинарном течении в эксцентрических каналах. Согласно Рейнольдсу и др. [1] мы определяем решения уравнения энергии, которые удовлетворяют определённым фундаментальным граничным условиям. Из этих фундаментальных решений можно сделать суперпозицию для того, чтобы удовлетворить целому ряду граничных условий. Точные решения уравнения энергии найти нельзя, поэтому получено приближенное решение. Числа Нуссельта, температуры и тепловые потоки на стенке, перепад температур стенка — жидкость и средние температуры жидкости представлены для широкого диапазона эксцентриситетов и отношений радиусов. Предполагается, что точность результатов составляет 1%.